Remarks

At this stage we can vigorously define a set to be *finite* if there exists a bijection of the set on to the set $\mathbf{N}_n = \{1, 2, 3,...,n\}$ for some natural number n. The *void* set ϕ is taken to be finite.

Example 1.5

If $A = \{1, 2, 3\}$ and f, g are relations corresponding to the subsets of $A \times A$ indicated against them, which of f, g is a function? Why?

$$f = \{(1, 3), (2, 3), (3, 2)\}$$

$$g = \{(1, 2), (1, 3), (3, 1)\}$$

Solution

f is a function since each element of A in the first place in the ordered pairs goes with only one element of A in the second place. g is not a function because 1 is related to both 2 and 3.

Example 1.6

If $f = \{(5, 2), (6, 3)\}$ $g = \{(2, 5), (3, 6)\}$, what is the range of f and g? Find fog.

Solution

$$\text{ran } f = \{2, 3\}, \text{ ran } g = \{5, 6\}$$

$$fog\,(2) = f(g(2)) = f(5) = 2$$

$$fog\,(3) = f(g(3)) = f(6) = 3$$

1.4 Binary Operations

A function $f: A \to A$, where A is a set can also be thought of as a *unitary operation* in the sense that an element of A is associated to each singleton subset $\{a\}$ of A. If an element of A is associated uniquely with every subset of two elements of A, the order of the elements being taken into account, we obtain a *binary operation* on A. More generally, a map $A \times A \to B$ is called a binary operation in A and if $B \subset A$, then it is said to be *closed* with reference to the operation. Formally, for $n = 1, 2,...$ an n-ary operation on the set A is a map $f: A \times A \times ... \times A$ (n times) $= A^n \to B$. For simplicity, we consider here unitary and binary operations only. If $A = \mathbf{R}^+$, the set of all positive real numbers, taking reciprocals, or, what is the same, the map

$x \to \frac{1}{x}$: $A \to A$ is a unitary operation. If $A = \mathbf{R}$, the set of all real numbers, $(x, y) = x+y$: $\mathbf{R}^2 \to \mathbf{R}$, or, what is the same, addition of two real numbers, is a binary operation on **R**. Multiplication is also a binary operation on **R**. However, division is not a binary operation on R, since division by 0 is not defined. But division is a binary operation on $\mathbf{R}\backslash\{0\}$. We can think of other binary operations in terms of these known binary operations. For instance, $(m, n) \to m+n+mn : \mathbf{Z}^2 \to \mathbf{Z}$, **Z** being the set of all integers, is a binary operation on **Z**. Keeping the map involved in the background, we prefer to speak of the binary operation '+' or addition instead of the map $(x,y) \to x+y$ or of the binary operation '.', or multiplication instead of the map $(x,y) \to xy$. With an analogous notation we can speak of the binary operation 'o' by writing $mon = m+n+mn$ on **Z**.

We pointed out that when defining a binary operation the order of the elements is to be taken into account; in other words, the map which defines the binary operation on A is on the set A^2 of all *ordered* pairs of elements of A. If o is the operation, for $a, b \in A$, aob and boa may be different elements of A. If, however, for every pairs a, b of elements A

$$aob = boa,$$

then the binary operation is *commutative*. For example, the binary operations +,., on **R** are commutative. Is the operation o defined at the end of the last paragraph commutative? The operation of division in $\mathbf{R}\backslash\{0\}$ is evidently *not* commutative. If the binary operation $*$ on **Z** is defined by

$$m*n = m-n+mn,$$

then $1*2 = 1-2+1.2 = 1$ while $2*1 = 2 - 1 + 2.1 = 3$ so that $1*2 \neq 2*1$ and $*$ is *not* a commutative binary operation.

If o is a binary operation on a set A and a, b, c are three elements of A, with due regard to the order in which a, b, c occur, we can consider the two elements

$$ao(boc), (aob)oc.$$

There is no *prima facie* reason for these to be one and the same. If, however, they are one and the same for *every* ordered triplet a, b, c of elements of A, then o is said to be an *associative* operation. As examples of associative binary operations, we have addition and multiplication defined on **R**. Clearly, division on $\mathbf{R}\backslash\{0\}$ is *not* an associative operation (Why?). The operation $*$ defined in the preceding paragraph which is not commutative is not also associative. If X is the set {a, b}, let the binary operation o be defined by $aoa = a$, $bob = b$, $aob = b$, $boa = a$. ...(1.1)
This operation is associative but *not* commutative (Verify). If the binary operation o is defined on **Z** by

$$lom = l+m-lm, \; l, m \in \mathbf{Z},$$

the operation o is evidently commutative. It is easily verified that this operation is *not*, however, associative. To sum up, the concepts of associativity and commutativity of a binary operation are independent.

If o is a binary operation on X and if there is $e \in X$ such that $aoe = eoa = a$, e is said to be an *identity element* for the operation. For instance, for the binary operation of addition in **R**, 0 is the identity element. For multiplication it is 1.

Remark

It is sometimes convenient to write down a binary operation by means of a table. For instance, the operation defined by(1.1)above is written in the form

o	a	b
a	a	b
b	a	b

The presence of one or more binary operations in a set gives a structure for the set which could be studied. For instance, we have because of the availability of such operations, the concepts of group, ring, vector space, field etc. The study of these structures is generally known as algebra.

Example 1.7

In the set N of natural numbers define the binary operation o by

$$m o n = \text{g.c.d}\ (m,n)\ ,\ m,n \in \mathbf{N}.$$

Is the operation commutative, associative?

Solution

The operation is clearly commutative since

$$\text{g.c.d}\ (m,n) = \text{g.c.d}\ (n,m)\ .$$

It is also associative because for $l,m,n \in \mathbf{N}$,

$$\text{g.c.d}\left(l,\ \text{g.c.d}\ (\text{m,n})\right) = \text{g.c.d}\left(\text{g.c.d(l,m)},\ \text{n}\right)$$

Example 1.8

Let $\mathbf{N}^k$ be the set of all ordered k-tuples of natural numbers. If $x = (x_1, \ldots, x_k)$, $y = (y_1,\ldots,y_k)$, $x_i, y_i \in \mathbf{N}$, $i = 1, 2, \ldots, k$ define $x+y = (x_1+y_1,\ldots, x_k+y_k)$. Then $+$ is a commutative and associative binary operation in $\mathbf{N}^k$.

Solution

It is easily verified that these properties are carried over from the corresponding properties of addition in N.

EXERCISE 1.1

1. Prove that $A^c \setminus B^c = B \setminus A$
2. Prove that $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$
3. If R is the relation "less than" from $A = \{1, 2, 3, 4, 5\}$ to $B\ \{1, 4, 5\}$,

write down the Cartesian product corresponding to R. Find the inverse relation to R.

4. If R is the relation in $\mathbf{N}\times\mathbf{N}$ defined by $(a,b)\ R\ (c,d)$ if and only if $a+d=b+c$, show that R is an equivalence relation.
5. If $\mathbf{N}_7=\{1,2,3,4,5,6,7\}$, which of the following two is a partition giving rise to an equivalence relation? Why?
   (i) $A_1=\{1,3,5\}$, $A_2=\{2\}$, $A_3=\{4,7\}$
   (ii) $B_1=\{1,2,5,7\}$, $B_2=\{3\}$, $B_3=\{4,6\}$
6. If $f:A\to B$, $g:B\to C$ are one-to-one or injective functions, show that gof is also one-to-one.
7. If $A=\{a,b,c,d\}$ and f corresponds to the Cartesian product $\{(a,b),(b,d),(c,a),(d,c)\}$, show that f is one-to-one from A onto A. Find f^{-1}.
8. Define the binary operation o in $\mathbf{N}$ by
   $mon=$ l.c.m. (m,n), $m,n\in\mathbf{N}$.
   Is the operation commutative and/or associative?
9. Does the table below give a commutative binary operation on the set $\{a,b,c\}$?

o	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

10. Establish the De Morgan's laws stated in section 1.1

11. Prove that there are 2^n elements in the class of all subsets of a set of n elements.

12. Give an example of a map (i) which is one-to-one but *not* onto, (ii) which is *not* one-to-one but onto, (iii) which is neither one-to-one *nor* onto.

13. If A is a non-empty set and $f,\ g:A\to A$ such that $fog=gof=I_A$, show that f and g are bijections and that $g=f^{-1}$

14. For any relation R in a set A, we can define the *inverse* relation R^{-1} by $a\ R^{-1}b$ if and only if bRa.
    Prove that
    (i) As a subset of $A \times A$, $R^{-1} = \{(b,a) \mid (a,b) \in R\}$
    (ii) R is symmetric if and only if $R = R^{-1}$
15. In $\mathbf{N} \times \mathbf{N}$, show that the relation defined by $(a,b)\ R\ (c,d)$ if and only if $ad = bc$ is an equivalence relation.
16. If $f: \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2-3x+2$, find $f(f(x))$.
17. If f, g: $\mathbf{R} \to \mathbf{R}$ are defined respectively by $f(x) = x^2+3x+1$, $g(x) = 2x-3$, find formulae for (i) fog, (ii) gof, (iii) fof, (iv) gog.
18. What is the set $\{x \mid x \in \mathbf{R}, x^2=9, 2x = 4\}$?
19. Is inclusion of a subset in another, in the context of a universal set, an equivalence relation in the class of subsets of the sets? Justify your answer.
20. How many relations are possible from a set A of m elements to another set B of n elements? Why?

CHAPTER 2

COMPLEX NUMBERS

2.1 The Algebra of Complex Numbers

The concept of numbers, as you are aware, is gradually extended from natural numbers to integers, from integers to rational numbers and from rational numbers to real numbers. You might have observed that this gradual extension is based on mathematical needs. For instance, there is no solution to the equation $x^2 = 2$, if $x \in \mathbf{Q}$, the set of rational numbers. Thus we are forced to introduce irrational numbers and thereby get the set $\mathbf{R}$ of the real numbers which is the union of $\mathbf{Q}$ with the set of all irrational numbers. We know that the square of every real number is non-negative, hence there exists no real number with its square equal to -1. In other words, there is no solution in $\mathbf{R}$ of the equation $x^2+1 = 0$. Just as we introduced the irrational number $\sqrt{2}$ so as to be able to solve the equation $x^2 = 2$, in the same way we assume that there is a number, which is not in $\mathbf{R}$, whose square is -1. This number is denoted by i. Thus $i^2=-1$ and i is a square root of -1. For any two real numbers a and b, we can form a new number $a+ib$. This number $a + ib$ is called a *complex number*. The set of all complex numbers is denoted by $\mathbf{C}$. If we write in the sequel $a+ib$ for an element of $\mathbf{C}$, it is tacitly assumed that a and b are real numbers unless otherwise stated. The real number a is called the real part of $a+ib$ and the real number b is called the *imaginary part*. We identify $a+i0$ with real number a and $i.1$ with i. A complex number is denoted by a single letter such as z, w and so on when it is convenient to do so.

The real part of a complex number z is denoted by Rez and the imaginary part as Imz.

If $z = x+iy$, where x and y are real, then Re$z = x$ and Im$z = y$.

We define addition and multiplication between two complex numbers as follows:

$$(a+ib) + (c+id) = (a+c) + i(b+d)$$

$$(a+ib)\,(c+id) = (ac-bd) + i\,(ad+bc)$$

It will be observed that while adding two complex numbers, the real and imaginary parts of the sum is obtained by adding separately the real and imaginary parts of the summands. As regards multiplication, we have multiplied the two complex numbers as if they were real numbers and have used the fact that $i^2 = -1$. If we denote $i(-1)$ by $-i$ as is usual, $(-i)(-i) = \big(0+i(-1)\big)\big(0+i(-1)\big) = -(-1)(-1) = -1$, so that $-i$ is also a square root of -1. Now $a+c$, $b+d$, $ac-bd$, $ad+bc$ are

obtained by using the binary operations of addition, multiplication, etc. in **R**. Thus in **C**, two binary operations + and · have been defined. These two operations are named addition and multiplication in **C**.

Let $z = a+ib$. If $b = 0$, then we say that z is a real number and is equal to a. If $a = 0$, then we say that z is a purely imaginary number and is equal to ib. We, more often, prefer to write bi such as $2i$, $3i$ etc. If both $a=0$ and $b = 0$, then instead of writing $0+i0$, we simply write 0. Two complex numbers $a+ib$ and $c+id$ are equal if and only if $a = c$ and $b = d$. In particular, $a+ib = 0$ if and only if $a = 0$ and $b = 0$.

We now consider the operation of addition in **C**. For any three complex numbers $a+ib$, $c+id$, $e+if$,

$$\begin{aligned}
&\{(a+ib) + (c+id)\} + (e+if)\\
&= \{(a+c) + i(b+d)\} + (e+if)\\
&= (a+c+e) + i(b+d+f)\\
&= (a+ib) + (c+e) + i(d+f)\\
&= (a+ib) + \{(c+id) + (e+if)\}
\end{aligned}$$

so that the operation is associative. It is also commutative, as is easily seen. We also see that

$$\begin{aligned}
(a+ib) + 0 &= (a+ib) + (0+i0) = a+0+i(b+0)\\
&= a+ib\\
&= (0+i0)+(a+ib)\\
&= 0 + (a+ib)
\end{aligned}$$

Thus 0 is an identity element for addition in **C**. If $z\in\mathbb{C}$, $z = a+ib$, we write $-z = -a+i(-b) = -a-ib$ and observe that

$$\begin{aligned}
z + (-z) = z-z &= (a+ib) + \{-a+i(-b)\}\\
&= 0+i0 = -z+z.
\end{aligned}$$

It is a good exercise to check that 0 is the only identity element for addition and that if $z+z' = 0$, then $z' = -z$. That $z+(-z) = -z+z = 0$ is expressed alternatively by saying that $-z$ is the additive inverse of z. Thus the binary operation of addition in **C** is associative, commutative, possesses identity element and is such that every element of **C** has additive inverse.

Let us now turn to multiplication. If $a+ib$, $c+id$, $e+if$ are any three complex numbers,

$$\begin{aligned}
(a+ib)\{(c+id)(e+if)\} &= (a+ib)\{(ce-df) + i(cf+de)\}\\
&= a(ce-df) - b(cf+de) + i\{a(cf+de) + b(ce-df)\}\\
&= ace-adf-bcf-bde+i(acf+ade+bce-bdf).
\end{aligned}$$

Likewise $\{(a+ib)(c+id)\}(e+if) = (ac-bd)e-(ad+bc)f+i\{(ac-bd)f+(ad+bc)e\}$

$= (ace-bde-adf-bcf)+i(acf-bdf+ade+bce)$

Thus,

$\{(a+ib)(c+id)\}(e+if) = (a+ib)\{(c+id)(e+if)\}$

Thus multiplication in C is associative. From the very definition, it is commutative. As said earlier, denoting $1+i0$ by 1, we see that for any complex number $a+ib$,

$1.(a+ib) = (a+ib).1 = (a+ib)(1+i0) = (a.1-b.0)+i(a.0+b.1) = a+ib.$

Similarly, it can be shown that $0.(a+ib) = (a+ib).0 = 0+i0 = 0$ for any complex number $a+ib$.

In other words, 1 is an identity element for the binary operation of multiplication in C and it is, in fact, the only one. For, if $a+ib \neq 0$ (this assumption may be made since $0.z=z.0=0$ for any complex number z)

and $(a+ib)(c+id) = a+ib$,

then $ac-bd = a,\ ad+bc = b$

i.e. $a(c-1) = bd,\ b(c-1) = -ad.$

Multiplying both sides of the first equation by a and of the second by b and adding, we get

$$(a^2+b^2)(c-1) = 0.$$

Since $a+ib \neq 0$, $a^2+b^2 \neq 0$, so $c = 1$. It turns out that $ad = 0$ and since a is not always 0, $d = 0$. So $c+id = 1$. Thus we see that the number 1 is unique. In this connection note that if k is any real number $k(a+ib) = ka+ikb$. In other words, multiplying a complex number z by a real number k amounts to multiplying the real and imaginary parts of z by k.

If $z = a+ib$ is a complex number, we denote $a+i(-b) = a-ib$ by $\bar{z}$ and call it the *conjugate* of z. It may be noted that the conjugate of $\bar{z}$ is z, i.e. $(\bar{\bar{z}}) = z$. If z is real, i.e. if $b = 0$, then evidently $z = \bar{z}$. Conversely, if $z = \bar{z}$, i.e. if $a+ib = a-ib$, then $b=-b$ so that $b = 0$.

To sum up, a complex number z is its own conjugate if and only if it is real. Since $z+\bar{z} = (x+iy)+(x-iy) = 2x$,

$$\text{Re } z = x = \frac{z+\bar{z}}{2}$$

Similarly, $\text{Im } z = y = \dfrac{z-\bar{z}}{2i}$

Also, if $z = a+ib$.

$z\bar{z} = (a+ib)(a-ib) = a^2+b^2 = \bar{z}z,$

since multiplication is commutative. Hence, we conclude that for any complex number z, the product $z\bar{z}$ is always a non-negative real number.

If $z \in C, z = a+ib$, a, b real, it is customary to denote the non-negative square root of a^2+b^2 by $|z|$, and call it the *modulus* or absolute value of the complex number z.

Thus,
$$z\bar{z} = |z|^2 \text{ and } |z| = \sqrt{a^2+b^2}$$
for any $z \in \mathbf{C}$. If $z \neq 0$, at least one of the two real numbers a and b is non-zero, so that $|z| \neq 0$. (In fact, $|z| = 0$ if and only if $z = 0$, as is easily verified). Thus if $z \neq 0$

$$z.\frac{\bar{z}}{|z|^2} = 1 = \frac{\bar{z}}{|z|^2}z.$$

If $z = a+ib \neq 0$, the complex number $\frac{\bar{z}}{|z|^2} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$

is again a complex number which is a multiplicative inverse of z, in the sense explained earlier. The inverse of z is denoted by z^{-1} or $\frac{1}{z}$. Thus $zz^{-1} = 1$. The uniqueness of the multiplicative inverse of $z \neq 0$ is easily verified. We know that a complex number $a+ib = 0$ if and only if $a = 0$ and also $b = 0$. This is equivalent to saying that $a+ib = 0$ if and only if $a^2+b^2 = 0$. Hence, $a+ib \neq 0$ if and only if $a^2+b^2 \neq 0$. In other words, $|z| = 0$ if and only if $z = 0$.

We are now in a position to define the binary operation of *division* in the set of complex numbers. As in the case of real numbers, this definition enables a complex number to be divided by a non-zero complex number. We define $\frac{z}{w}$ as $z.w^{-1}$. Thus if $z = x+iy$, $w = u+iv$, $u+iv \neq 0$, i.e. $u^2+v^2 \neq 0$, then

$$\frac{z}{w} = z.w^{-1} = (x+iy).\frac{\bar{w}}{|w|^2} = (x+iy)\left(\frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}\right)$$
$$= \left(\frac{xu+yv}{u^2+v^2}\right) + i\left(\frac{yu-xv}{u^2+v^2}\right)$$

In particular, $\frac{1}{w} = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$ as was seen earlier.

We can now prove the following simple and important result. If z and w are two complex numbers such that their product $zw = 0$, then at least one of the numbers z, w is zero. Suppose, as we may, that $z \neq 0$. Then z^{-1} exists. Now $zw = 0$. So $z^{-1}(z.w) = 0$. By associativity of multiplication $(z^{-1}z)w = 0$ or $1.w = 0$ i.e., $w = 0$. Thus if $z \neq 0$, then w must be equal to zero, otherwise $z = 0$. As a corollary of this result we can say that the product of two non-zero complex numbers is not zero. That is, if $z \neq 0$, $w \neq 0$, then $zw \neq 0$.

Besides these properties, we have the following which connect up addition and multiplication:

If $z_1, z_2, z_3 \in \mathbf{C}$, then
$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3,$$
$$(z_1+z_2)\ z_3 = z_1\ z_3 + z_2\ z_3$$

These properties are known as *distributive properties* and signify that multiplication distributes over addition. These properties are verified easily from the definition of addition and multiplication in C. Addition and multiplication are related by the

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distributive laws, making **C** what is familiarly known as a *field*. Note that **R** too is a field. Every complex number $a+i0$ can be identified as stated earlier with the real number a. It can also be seen that $(a+i0)+(b+i0) = a+b+i0$ and $(a+i0)(b+i0) = ab+i0$. Hence, the sum and product of complex numbers $a+i0$ and $b+i0$ can be identified with the sum and product of the real numbers a and b. Hence **R**, the set of real numbers, can be considered to be a subset of **C**, the set of complex numbers.

It is clear that $\mathbf{R} \neq \mathbf{C}$ in the sense that there are elements of **C** which cannot be identified with elements of **R** in the identification mentioned above. For instance, $0+i.1 \in \mathbf{C}$ and it is not equal to $a+i.0$ for any $a \in \mathbf{R}$. Thus **R** can be deemed as a proper subset of **C**.

You may recall that in defining addition and multiplication of complex numbers, we have used the fact that $i^2 = -1$. Also $(-i)^2 = -1$. However, i being a symbol, it is immaterial which you call i and which you call $-i$.

For two complex numbers z_1 and z_2, we have

$$\overline{(z_1+z_2)} = \bar{z}_1+\bar{z}_2;\ \overline{(z_1-z_2)} = \bar{z}_1-\bar{z}_2$$

and

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2;\ \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \text{ where } z_2 \neq 0.$$

These results can be verified directly by setting $z_1 = x_1 + iy_1$ and $z_2 = x_2+iy_2$.

Let us recall that for a complex number z, $z\bar{z} = |z|^2$. Now, for two complex numbers z_1 and z_2, we have

$$|z_1z_2|^2 = (z_1z_2)\,\overline{(z_1z_2)} = z_1z_2\bar{z}_1\bar{z}_2$$
$$= z_1\bar{z}_1.z_2\bar{z}_2 = |z_1|^2.|z_2|^2$$

Taking square root and noting that positive sign can be taken on both sides, we get $|z_1z_2| = |z_1|.|z_2|$.

The above result is expressed in words as follows:

The absolute value of a product of two complex numbers is equal to the product of the absolute values of the numbers.

If $z_2 \neq 0$, then $z_1 = \left(\frac{z_1}{z_2}\right) z_2$ and so

$$|z_1| = \left|\frac{z_1}{z_2}\right| \cdot |z_2|$$

Hence,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Before we give a formula for the absolute value of the sum of two complex numbers, let us recall that the real and imaginary parts of a complex number $z = x+iy$ are given by $\text{Re } z = \frac{z+\bar{z}}{2}$ and $\text{Im}z = \frac{z-\bar{z}}{2i}$. Hence, $2\text{Re } z = z+\bar{z}$, and $2i\text{Im}z = z-\bar{z}$. Now, for two complex numbers z_1 and z_2.

$$|z_1+z_2|^2 = (z_1+z_2)(\bar{z}_1+\bar{z}_2) = z_1\bar{z}_1+z_2\bar{z}_2+z_1\bar{z}_2+\bar{z}_1z_2$$
$$= |z_1|^2 + |z_2|^2+z_1\bar{z}_2+\overline{(z_1\bar{z}_2)}$$
$$= |z_1^2+|z_2|^2 + 2 \text{ Re } z_1\bar{z}_2 \quad \ldots(2.1)$$

We have also

$$|z_1-z_2|^2 = (z_1-z_2)(\bar{z}_1-\bar{z}_2) = z_1\bar{z}_1 + z_2\bar{z}_2 - (z_1\bar{z}_2+\bar{z}_1z_2)$$
$$= |z_1|^2 + |z_2|^2 - 2\text{Re}(z_1\bar{z}_2) \quad \ldots(2.2)$$

Adding (2.1) and (2.2) we get

$$|z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2) \quad \ldots(2.3)$$

Again $x^2 \leq x^2+y^2$ and $y^2 \leq x^2+y^2$ and taking positive square roots, we get $|x| \leq |z|$ and $|y| \leq |z|$ leading to the conclusion that $-|z| \leq |x| \leq |z|$ and $-|z| \leq y \leq |z|$. The equality $x = \text{Re } z = |z|$ holds if and only if z is real and non-negative. For Re $z = |z|$ implies $x^2 = x^2+y^2$ if $z = x + iy$, so that $y = 0$.
Using the inequality $-|z| \leq \text{Re } z \leq |z|$ in (2.1), we get

$$|z_1+z_2|^2 \leq |z_1|^2+|z_2|^2+2|z_1||z_2| = (|z_1|+|z_2|)^2,$$

(since $|\bar{z}_2| = |z_2|$)

Taking positive square roots, we obtain

$$|z_1 + z_2| \leq |z_1|+|z_2| \quad \ldots(2.4)$$

(2.4) is called *triangle inequality*. The reasons for this nomenclature will be explained later.

Of the two complex numbers z_1 and z_2, let $|z_1| \geq |z_2|$.
Now,

$$z_1 = z_1-z_2+z_2, \text{ i.e. } |z_1| \leq |z_1-z_2|+|z_2|$$

Thus, $|z_1|-|z_2| \leq |z_1-z_2|$.
By interchanging z_1 and z_2, we get
$|z_2|-|z_1| = -(|z_1|-|z_2|) \leq |z_1-z_2|$ or equivalently $|z_1|-|z_2| \geq -|z_1-z_2|$
Combining the two inequalities, we get

$$||z_1|-|z_2|| \leq |z_1-z_2| \quad \ldots(2.5)$$

for any two complex numbers z_1 and z_2. The inequality (2.5) is also called triangle inequality. The inequality (2.4) can be extended to n complex numbers by finite induction which will be discussed later. In other words.

$$|z_1+z_2+\ldots+z_n| \leq |z_1|+|z_2|+\ldots+|z_n| \quad \ldots(2.6)$$

for any n complex numbers $z_1, z_2, \ldots z_n$.

Example 2.1

Express (i) $(-1+2i)+(\frac{1}{2}-i)$

(ii) $(\frac{1}{2}+\frac{1}{4}i)(-\frac{2}{3}-\frac{1}{4}i)$

in standard form.

Solution

(i) $(-1 + 2i) + (\frac{1}{2} - i)$

$= (-1 + \frac{1}{2}) + (2-1)i$ (by definition)

$= -\frac{1}{2} + 1i = -\frac{1}{2} + i$

(ii) By definition,

$(\frac{1}{2} + \frac{1}{4}i)(-\frac{2}{3} - \frac{1}{4}i)$

$= (-\frac{1}{3} + \frac{1}{16}) + (-\frac{1}{8} - \frac{1}{6})i$

$= -\frac{13}{48} - \frac{7}{24}i$

Remark

When a, b are specific real numbers, we prefer to write $a + bi$ to $a+ib$, e.g. we write $2+3i$ instead of $2+i3$. Further, we prefer to write $1.i$ as i.

Example 2.2

Show that for any real θ, $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = 1$

Solution

$(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)$
$= (\cos^2\theta + \sin^2\theta) + i(-\cos\theta\sin\theta + \sin\theta\cos\theta)$
$= 1 + 0.i = 1$, since $\cos^2\theta + \sin^2\theta = 1$ for any θ.

Remark

This example shows that if $z = \cos\theta + i\sin\theta$, then $\frac{1}{z} = \cos\theta - i\sin\theta$. If we denote $\cos\theta + i\sin\theta$ by $e^{i\theta}$, this means $\frac{1}{e^{i\theta}} = e^{i(-\theta)}$ which we may write as $e^{-i\theta}$

Example 2.3

Show that $\overline{(2+3i)(1-i)} = (2-3i)(1+i)$

Solution

$\overline{(2+3i)(1-i)} = \overline{(2+3) + (-2+3)i} = \overline{5+i} = 5-i$
Again, $(2-3i).(1+i) = (2+3) + (2-3)i = 5-i$, proving the equality.

[1]*Remark*

The example suggests that if a complex number is given in terms of sums, products or quotients of complex numbers in standard form (i.e. of the form $a+ib$ where a, b are real), then its conjugate is obtained by changing i to $-i$ in all the constituents.

Example 2.4

Find the reciprocal or inverse of $1-i$.

Solution

$$\frac{1}{1-i} = \frac{1+i}{(1-i)(1+i)} = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i.$$

EXERCISE 2.1

1. If $z_1, z_2 \in \mathbf{C}$, show that $(z_1+z_2)^2 = z_1^2+2z_1z_2+z_2^2$
2. Show that $(1-i)^2 = -2i$
3. Find the conjugate of $\frac{1-i}{1+i}$
4. Show that if $a, b, c, d \in \mathbf{R}$, $(a+ib)\,(c+id) = (a-ib)\,(c-id)$

2.2 The Argand Diagram and the Polar form

One can think of another notation for a complex number. Instead of writing $a+ib$, a, b real, we could write an ordered pair (a,b) of real numbers as standing for a complex number. We can rewrite the definition of addition and multiplication as follows:

$$(a, b) + (c, d) = (a+c, b+d)$$
$$(a, b).(c, d) = (ac-bd, ad+bc)$$

and develop whatever was done in section 2.1 in this notation. This notation is suggestive of the fact that a complex number can be represented by a point (a, b) in a plane referred to a pair of rectangular axes (as in analytical geometry which you will study a little later). Let $x'Ox$, $y'Oy$ be two perpendicular lines in a plane, say, the plane of this paper. If P is a point in this plane, drop the perpendiculars from P on $x'Ox$ and $y'Oy$ with feet A, B respectively. Note that $BPAO$ is a rectangle. If the length of OA is a units and that of OB is b units, and A is to the right of O while B is above O, we say that the x-coordinate of the point P is a and the y-coordinate of P is b. We also say

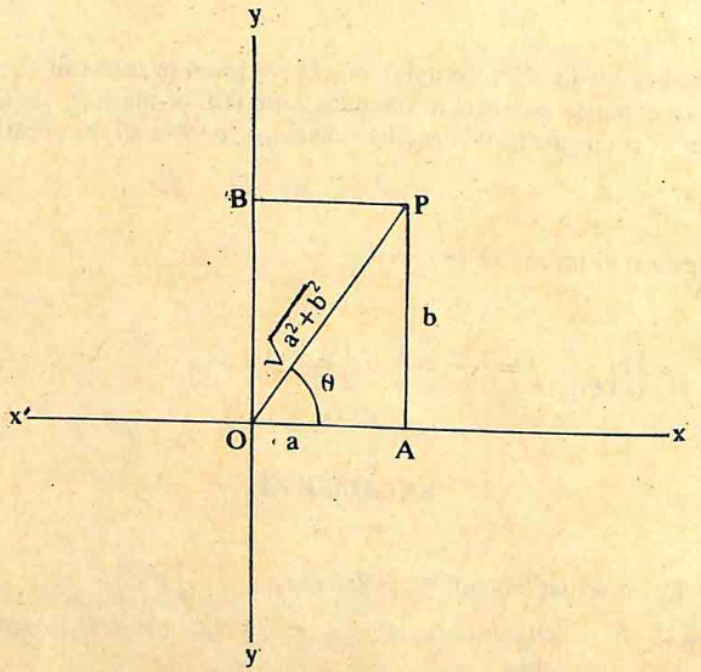

Fig. 2.1

that P is the point (a, b). If A is a units to the left of O while B is b units above O, then P is the point $(-a, b)$. Likewise depending on whether B is above or below O, the second coordinate b has a positive or negative sign before it. Note that the signs of a and b determine the direction of OP and in this sense the coordinates of P denote the tip P of the directed line segment or vector $\overrightarrow{OP}$ in the plane. Note that $OP = \sqrt{a^2+b^2}$. If a complex number $z = a+ib$, then the point P with coordinates (a, b) is taken to represent the complex number $a+ib$. Since $OP = \sqrt{a^2+b^2}$, we have $|z| = OP$. This representation of complex numbers as points in the plane is known as the *Argand diagram*. The plane is called the *complex plane*. Note that only one complex number corresponds to a given point in the Argand diagram and vice versa. Thus we have a bijection between $\mathbf{C}$ and $\mathbf{R}^2$ $(= \mathbf{R} \times \mathbf{R})$.

Note that if $z \in \mathbf{C}$ is represented by P in the Argand diagram, $\bar{z}$, the conjugate of z is represented by $\bar{P}$ which is symmetric to P with respect to the axis $x'Ox$ or $\bar{P}$ is the (mirror) reflexion of P on the axis $x'Ox$. (See Fig. 2.2).

It is customary to call the axis $x'Ox$ the *real axis* and $y'Oy$ the *imaginary axis* for the reason that points on $x'Ox$ correspond to real numbers while those on $y'Oy$ to purely imaginary numbers.

Let P_1, P_2 represent two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ respectively in the Argand diagram in the complex plane as shown in figure 2.3. Join the origin O with the points P_1 and P_2 respectively and complete the parallelogram by drawing a line parallel to OP_1 from P_2 and another parallel to OP_2 from P_1, which intersect at the point P. It is clear from figure 2.3 that the coordinates of P are (x_1+x_2, y_1+y_2).

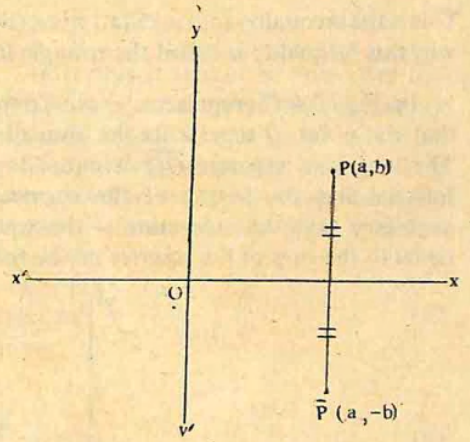

Fig. 2.2

Hence, the point P represents the complex number $(x_1+x_2) + i(y_1+y_2) = z_1+z_2$. The length of the diagonal $\overline{OP}$ of the parallelogram OP_1PP_2 or the distance of P from O is equal to $|z_1+z_2|$. Now the segment $\overline{OP_2}$ = the segment $\overline{P_1P}$. Hence, the length of each of them is equal to $|z_2|$. We know from school geometry that the sum of two sides of a triangle is greater than the third side. Hence, in ΔOPP_1, $\overline{OP} \leq \overline{OP_1} + \overline{P_1P}$.

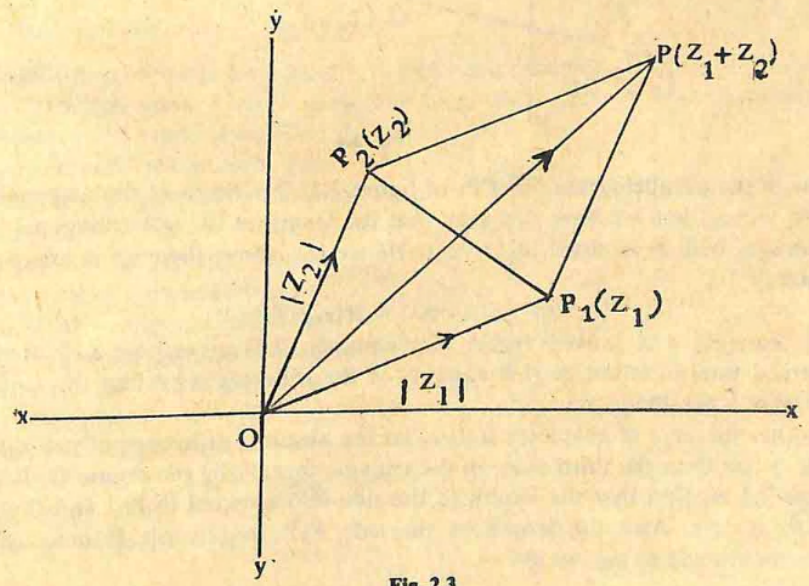

Fig. 2.3

This is the inequality $|z_1+z_2| \leq |z_1|+|z_2|$ (See inequality (2.4) of sec. 2.1). This is the reason why this inequality is called the triangle inequality.

In Fig. 2.4 P_2' represents $-z_2$. Completing the parallelogram $OP_2'QP_1$, we find that the point Q represents the sum of the complex numbers z_1 and $-z_2$ or z_1-z_2. The length of segment $\overline{OQ}$ is equal to $|z_1-z_2|$. Since OQ = the segment $\overline{P_1P_2}$, it follows that the length of the segment $\overline{P_1P_2}$ is equal to $|z_1-z_2|$. A theorem in geometry says that the sum of the squares on the diagonal of a parallelogram is equal to the sum of the squares on the four sides.

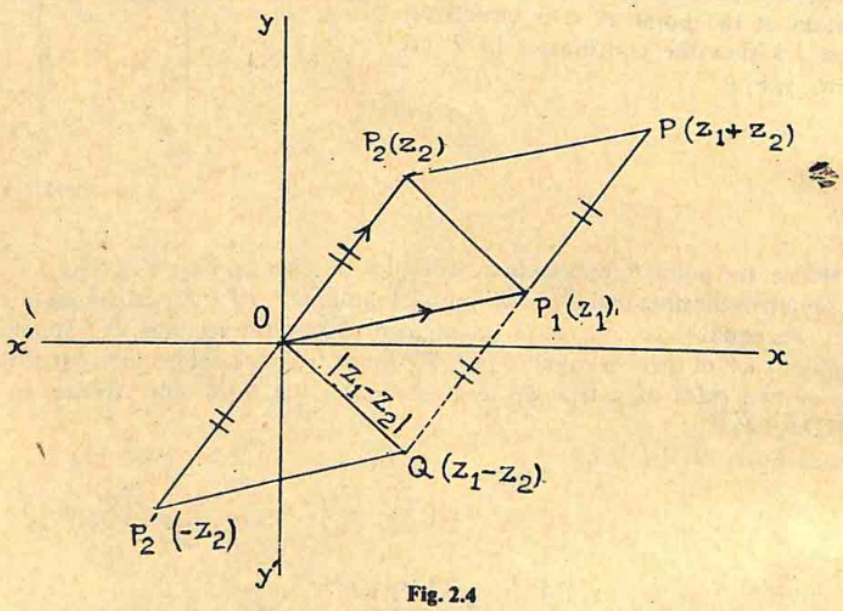

Fig. 2.4

Now in the parallelogram OP_1PP_2 of figure 2.3, the length of the diagonal OP is equal to $|z_1+z_2|$ and we have just seen that the length of the other diagonal joining the points z_1 with z_2 is equal to $|z_1-z_2|$. Hence, the above theorem is expressed by the equality

$$|z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

This equality was proved earlier (See equality (2.3) of section 2.1). Hence the geometrical interpretation of this equality is the theorem regarding the squares of diagonals of a parallelogram.

Another theorem of geometry states that the absolute difference of two sides of a triangle is less than the third side. In the triangle formed by the points O, P_1 and P_2 in figure 2.3 we find that the length of the side $\overline{OP_1}$ is equal to $|z_1|$ and that of the side $\overline{OP_2}$ is $|z_2|$. Also the length of the side P_2P_1 is $|z_1-z_2|$. Hence, using the theorem mentioned above, we get

$$\big||z_1|-|z_2|\big| \leq |z_1-z_2|.$$

This is the inequality (2.5) of section 2.1. Conversely it should be now clear that the geometrical interpretation of the above inequality is the theorem stating that the absolute difference of two sides of a triangle is less than the third side.

Inequality (2.5) of section 2.1 is also called *triangle inequality*. In fact it is deducible from the triangle inequality mentioned earlier and yields that inequality too. This means that they are logically equivalent. For instance, writing $z_1+z_2=z$, $z_2=w$, we have $z_1 = z-w$ and triangle inequality yields

$$|z| \leq |z-w|+|w|$$

or

$$|z-w| \geq |z|-|w| \qquad \text{(A)}$$

Since z, w are just symbols, interchanging z and w, we have

$$|z-w| \geq |w|-|z| = -\big(|z|-|w|\big) \qquad \text{(B)}$$

$|z|-|w|$ is real and so results (A) and (B) imply

$$|z-w| \geq \big|\,|z|-|w|\,\big|$$

for any two complex numbers z and w. Writing $z=z_1$, $w=z_2$, we have

$$\big||z_1|-|z_2|\big| \leq |z_1-z_2|,$$

for any two complex numbers z_1, z_2

2.3 Polar Representation

The introduction of polar coordinates to represent complex numbers in the plane is helpful in various ways. First, it is helpful in giving a geometrical interpretation of the product of two complex numbers. Second, the powers and roots of complex numbers can be very conveniently defined with the help of their polar representations. Let P represent in the complex plane a non-zero complex number $z = x+iy$ and let the directed line OP make with the positive direction of the x-axis an angle θ. θ is measured in radians. Note that 2π radians are 360°. If we denote the length of the segment OP by r, then $r = \sqrt{x^2+y^2} = |z|$ and $x = r\cos\theta$, $y = r\sin\theta$.

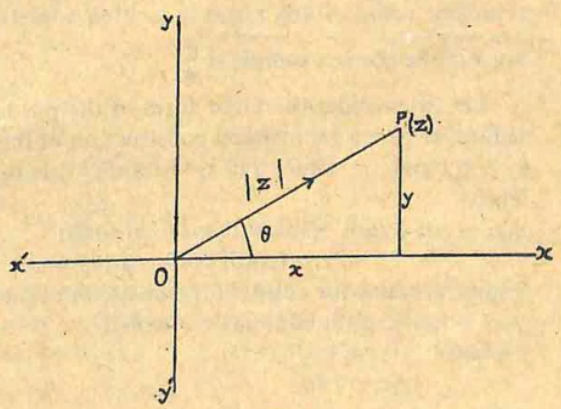

Fig. 2.5

and $\tan\theta = \frac{y}{x}$. (r, θ) are called the polar coordinates of the complex number z. Clearly, $r \geq 0$, and as θ, the angle POx is measured in radians, $0 \leq \theta \leq 2\pi$. For every positive value of r and each value of θ between 0 and 2π, we get a unique point in the complex plane with polar coordinates as (r, θ) and conversely, every point in the plane has polar coordinates (r, θ), where $r \geq 0$ and $0 \leq \theta \leq 2\pi$. The segment $\overline{OP}$ is called the radius vector of the point P. The number $r = 0$ if and only if $z = 0$. This is in keeping with r being the modulus of z and θ is called its *argument* or *amplitude*. We write $\arg z = \theta$

Example 2.5

Represent the complex number $1+i$ in the polar form.

Solution

Let (r, θ) be the required coordinates. Then $r\cos\theta = 1$ and $r\sin\theta = 1$. Hence, $r^2(\cos^2\theta + \sin^2\theta) = 2$, or $r = \sqrt{2}$ (r is always ≥ 0) and $\cos\theta = \frac{1}{\sqrt{2}}$, $\sin\theta = \frac{1}{\sqrt{2}}$ and $\tan\theta = 1$. Hence, $\theta = \tan^{-1} 1 = \pi/4$.

Remark

In this example, θ can be taken to be equal to $\frac{\pi}{4} + 2\pi$, $\frac{\pi}{4} + 4\pi$, and so on. You will learn later that $\tan(\theta + 2n\pi) = \tan\theta$. If we restrict θ to be lying between 0 and 2π, then there will be only one value of θ. It must, therefore, be noted that the value of θ is determined up to a multiple of 2π. Thus in the above example $\theta = \frac{\pi}{4} + 2n\pi$, $n = 0, \pm 1, \pm 2, \pm 3 \ldots$ That value of θ which is such that $-\pi < 0 \leq \pi$ is called the *principal value* of $\arg z$ and is written sometimes as $\text{Arg } z$. The principal value of $\arg z$ in the above example is $\frac{\pi}{4}$.

Let us consider the polar form of the product of two complex numbers z_1 and z_2 before we give a geometrical construction of their product z_1z_2. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Then

$$z_1z_2 = r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)$$

From formulae for $\cos(\theta_1+\theta_2)$ and $\sin(\theta_1+\theta_2)$ which you will learn later, we have

$$z_1z_2 = r_1r_2\{\cos(\theta_1+\theta_2) + i(\sin(\theta_1+\theta_2)\} \quad \ldots (2.7)$$

Hence,

$$|z_1z_2| = r_1r_2$$

and

$\arg(z_1z_2) = \theta_1+\theta_2 = \arg z_1+\arg z_2$ .... (2.8)

If we write $z_1 = r_1e^{\theta_1}$, $z_2 = r_2e^{i\theta_2}$, (2.8) shows that $z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$

Here the relation (2.8) is to be interpreted in the following manner:

Any agrument of z_1z_2 can be expressed as any argument of z_1 plus some argument of z_2 and conversely, any argument of z_1 plus any argument of z_2 is equal to an argument of z_1z_2.

It may be remarked that equation (2.8) may not be valid if we use 'Arg' instead of 'arg', i.e. if we take principal values.

Now we give a geometrical representation of the product z_1z_2. We will use relations (2.7) and (2.8) for this purpose. Let $z_1 = r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2+i\sin\theta_2)$.

Let the points P_1 and P_2 represent the complex numbers z_1 and z_2. In Fig. 2.6, O is the origin and I is the point representing the number 1 on the real axis. Then the points O, P_1, I are the vertices of a triangle viz ΔOP_1I. Denote by ψ the angle OIP_1 as shown in Fig. 2.6.

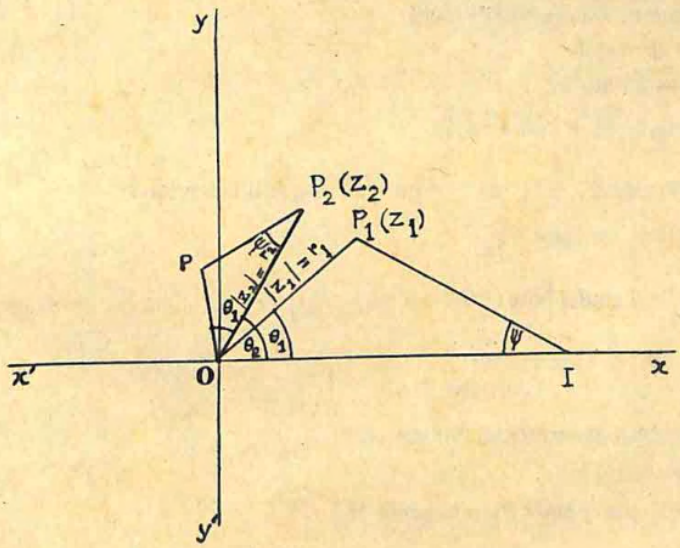

Fig. 2.6

Draw a line OP from O making angle equal to θ_1 with the segment OP_2, and another line from the point P_2 making an angle equal to ψ with the segment P_2O. intersecting the line OP at the point P. Then the point P represents the complex number z_1z_2.

To prove this we note that ΔOP_1I is similar to ΔOPP_2, since $\angle P_1OI = \angle POP_2 = \theta_1$ and $\angle P_1IO = \angle PP_2O = \psi$. Thus the three angles of one triangle are respectively equal to the three angles of the other triangle. Hence,

$\frac{OP}{r_1} = \frac{r_2}{1}$ Thus

$OP = r_1r_2$ and $\angle POI = \theta_1+\theta_2$.

Hence, P represents the complex number $r_1r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$. It follows from relations (2.7) and (2.8) that P represents the complex number z_1z_2.

Before giving a construction for $\frac{z_1}{z_2}$ $(z_2 \neq 0)$, we observe that for $z \neq 0$,

$$\frac{1}{z} = \frac{1}{r(\cos\theta+i\sin\theta)}$$

$$\text{i.e. } \frac{1}{z} = \frac{\cos\theta - i\sin\theta}{r(\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)}$$

$$\text{i.e. } \frac{1}{z} = \frac{\cos\theta - i\sin\theta}{r(\cos^2\theta+\sin^2\theta)}$$

$$\text{So } \frac{1}{z} = \frac{1}{r}\{\cos(-\theta) + i\sin(-\theta)\}$$

since $\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$, as you will learn later.

Hence, for $z_2 \neq 0$, we have

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos\theta_1 + i\sin\theta_1)\{\cos(-\theta_2) + i\sin(-\theta_2)\}$$

$$= \frac{r_1}{r_2}\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2).\}$$

The corresponding geometrical construction is given in Fig. 2.7.

In Fig. 2.7, the point P represents the complex number $\frac{z_1}{z_2}$. In the two similar triangles

$$\frac{OP}{1} = \frac{OP_1}{OP_2} = \frac{|z_1|}{|z_2|} = \frac{r_1}{r_2}$$

and the line OP makes with the real axis an angle equal to $\theta_1 - \theta_2$.

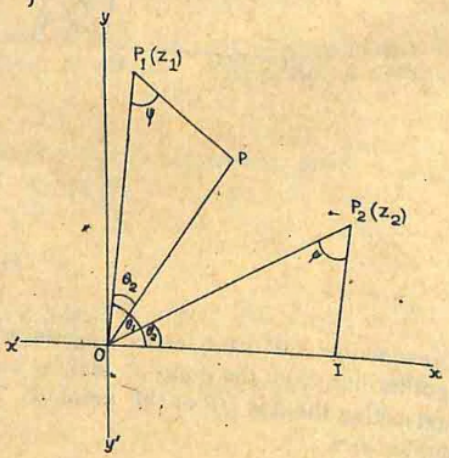

Fig. 2.7

2.4 Powers and Roots of Complex Numbers

If $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$, then it follows from relation 2.7 of section 2.3 that

$$z_1 z_2 = r_1 r_2 \{\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)\}$$

If $z_3 = r_3 (\cos \theta_3 + i \sin \theta_3)$ is another complex number, then it follows that

$$\begin{aligned} z_1 z_2 z_3 &= r_1 r_2 \{\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)\}\, r_3 (\cos \theta_3 + i \sin \theta_3) \\ &= r_1 r_2 r_3 \{\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)\} (\cos \theta_3 + i \sin \theta_3) \\ &= r_1 r_2 r_3 \{\cos (\theta_1 + \theta_2 + \theta_3) + i \sin (\theta_1 + \theta_2 + \theta_3)\} \end{aligned}$$

for the same reason.

Using induction, it can be shown that the above formula can be extended to an arbitrary product of finite number of complex numbers. That is, if $z_1, z_2 \ldots, z_n$ are n complex numbers, then

$$z_1 z_2 \ldots z_n = r_1 r_2 \ldots r_n \{\cos (\theta_1 + \theta_2 + \ldots + \theta_n) + i \sin (\theta_1 + \theta_2 + \ldots + \theta_n)\} \quad \ldots \textbf{(2.9)}$$

Taking $z_1 = z_2 = z_3 = \ldots = z_n = z$, and $\theta_1 = \theta_2 = \ldots = \theta_n = \theta$, and $r_1 = r_2 = \ldots = r_n = r$ in (2.9), we get

$$z^n = r^n (\cos n\theta + i \sin n\theta), \; n = 1, 2, 3 \quad \ldots \textbf{(2.10)}$$

Taking $r = 1$, and $z = \cos \theta + i \sin \theta$ in (2.10) we have

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta, \text{ for } n = 1, 2, 3. \quad \ldots \textbf{(2.11)}$$

(2.11) gives the important *Laplace-de Moivre's formula* for positive integral exponents n. The above formula (2.11) is evidently true for $n = 0$. Also, since

$$z^{-1} = \frac{1}{z},$$

$$(\cos \theta + i \sin \theta)^{-1} = \frac{1}{\cos \theta + i \sin \theta}$$

$$= \frac{\cos \theta - i \sin \theta}{(\cos \theta + i \sin \theta)(\cos \theta - i \sin \theta)}$$

$$= \cos (-\theta) + i \sin (-\theta) \quad \ldots \textbf{(2.12)}$$

Hence, formula (2.11) is valid for $n = -1$. Let n be a negative integer, say $n = -m$, where $m > 1$. Then

$$(\cos \theta + i \sin \theta)^n = (\cos \theta + i \sin \theta)^{-m} = \{(\cos \theta + i \sin \theta)^{-1}\}^m$$
$$= \{\cos (-\theta) + i \sin (-\theta)\}^m \quad \text{[from (2.12)]}$$

Hence

$$(\cos \theta + i \sin \theta)^n = (\cos \theta + i \sin \theta)^{-m} = \{\cos (-\theta) + i \sin (-\theta)\}^m$$
$$= (\cos (-m\theta) + i \sin (-m\theta)), \text{ since } m \text{ is a positive integer.}$$

Therefore, (2.11) holds for $n = -m$, m a positive integer.

In other words,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta, \text{ for } n = \pm 1, \pm 2, \pm 3, \ldots \quad \ldots \textbf{(2.13)}$$

Thus, the Laplace-de Moivre's formula is valid for all integral values of n, positive, zero, and negative.

This formula can be extended to any real number (rational or irrational), but proving that is beyond the scope of this book.

We are now in a position to find the nth root of a given complex number a for a positive integer n. Let

$$a = \rho(\cos\psi + i\sin\psi)$$

As the case $n = 1$ is trivial, we take $n \geq 2$. We have to find a complex number $z = r(\cos\theta + i\sin\theta)$ such that

$$z^n = a$$

i.e. $r^n(\cos n\theta + i\sin n\theta) = \rho(\cos\psi + i\sin\psi)$, ... **(2.14)**

If we take $r^n = \rho$ and $n\theta = \psi$, the equation (2.14) will definitely be satisfied. Thus $r^n = \rho$ and $\theta = \frac{\psi}{n}$.

Hence we get a root

$z = \sqrt[n]{\rho}\left(\cos\frac{\psi}{n} + i\sin\frac{\psi}{n}\right)$, where $\sqrt[n]{\rho}$ denotes the positive nth root of the positive real number ρ. But the equation (2.14) is valid, as observed earlier when we add an integral multiple of 2π to ψ. That is to say that

$$n\theta = \psi + k.2\pi, \text{ where } k \text{ is an integer}$$

i.e. $$\theta = \frac{\psi}{n} + k.\frac{2\pi}{n}$$

Hence,

$$z = \sqrt[n]{\rho}\left[\cos\left(\frac{\psi}{n} + k.\frac{2\pi}{n}\right) + i\sin\left(\frac{\psi}{n} + k.\frac{2\pi}{n}\right)\right] \quad \text{... (2.15)}$$

where $k = 0, 1, 2, \ldots(n-1)$,

for, only these values of k will give different values of z and other integral values, positive or negative, of k will only repeat the values obtained by taking $k = 0, 1, 2, \ldots (n-1)$. Note further that if $\sqrt[n]{\rho}(\cos\theta_1 + i\sin\theta_1)$ is an nth root of a, then $\theta_1 = \psi_1 + \frac{2k\pi}{n}$ for some $k = 0, 1, 2, \ldots, n-1$, where $\psi_1 = \frac{\psi}{n}$

Thus, the number of the nth root of a non-zero complex number is n and the modulus of each of these roots is the same non-negative real number. The arguments of these n roots are equally spaced in the sense that if ψ is the principal value of arg a i.e. $-\pi < \psi \leq \pi$, then the arguments of other roots of a are obtained by adding respectively $\frac{2\pi}{n}, \frac{4\pi}{n} \ldots$ to $\frac{\psi}{n}$

The root, having its argument in the interval $(-\pi, \pi)$ is called the *principal root* of a.

Let us apply this result to find the square root of a complex number $a = \rho(\cos\psi + i\sin\psi)$,

$-\pi<\psi\leq\pi$. Using the result (2.15), the required square roots of a are $\sqrt{\rho}\left(\cos\frac{\psi}{2}+i\sin\frac{\psi}{2}\right)$ and $\sqrt{\rho}\left\{\cos\left(\frac{\psi}{2}+\pi\right)+i\sin\left(\frac{\psi}{2}+\pi\right)\right\}$. The second root can be written as $-\sqrt{\rho}\left(\cos\frac{\psi}{2}+i\sin\frac{\psi}{2}\right)$.

If $a=1$, then $\rho=1$ and $\psi=0$. Hence the two square roots are 1 and -1.

The cube roots and fouth roots of 1 are of special interest. Since $1=1\times(\cos 0+i\sin 0)$, the three cube roots of 1 are $\sqrt[3]{1}\left(\cos\frac{0}{3}+i\sin\frac{0}{3}\right)$, $\sqrt[3]{1}\left[\cos\left(\frac{0}{3}+\frac{2\pi}{3}\right)+i\sin\left(\frac{0}{3}+\frac{2\pi}{3}\right)\right]$ and $\sqrt[3]{1}\left[\cos\left(\frac{0}{3}+\frac{4\pi}{3}\right)+i\sin\left(\frac{0}{3}+\frac{4\pi}{3}\right)\right]$. Hence the three cube roots of 1 are $1, -\frac{1}{2}+i\frac{\sqrt{3}}{2}$ and $-\frac{1}{2}-i\frac{\sqrt{3}}{2}$. All these three roots lie on the circumference of the unit circle, as shown in Fig 2.8, and the angles between the radii vector of the first and the second, the second and the third, and the third and the first are each $\frac{2\pi}{3}$ radians or 120^0. Hence, if these points

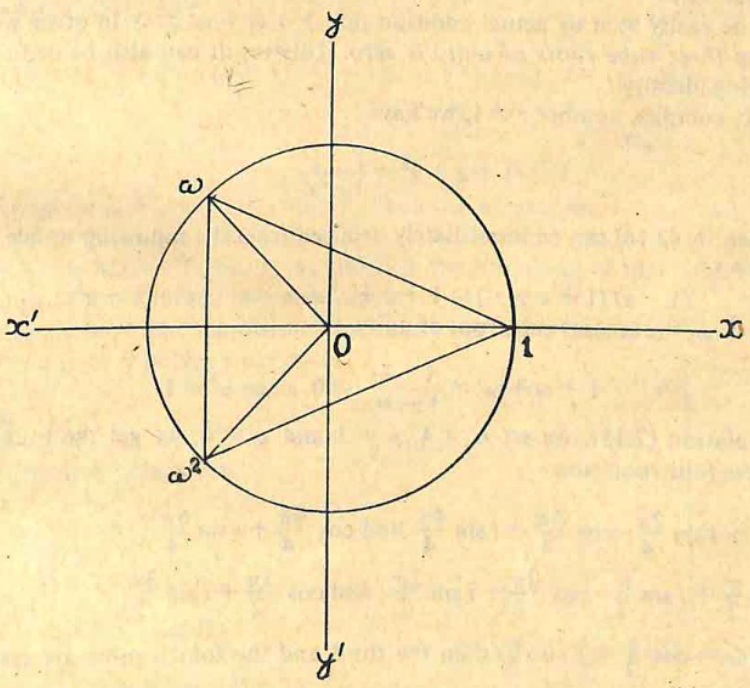

Fig. 2.8

are joined by straight lines, they will form the three vertices of an equilateral triangle. If we denote the second root by ω i.e. $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, the third root will be ω^2 as can be seen by actual computation, since

$$\omega^2 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 = \left(\frac{1}{4} - \frac{3}{4} - 2i\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)$$
$$= -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

which is the third root. Hence the three roots are 1, ω and ω^2.

The fact that the third root is the square of the second root can also be seen without any computation, because $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ and the third root is $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$ which can be written as $\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^2$ by using Laplace-de Moivre's formula.

It can be easily seen by actual addition that $1 + \omega + \omega^2 = 0$. In other words, *the sum of the three cube roots of unity is zero.* This result can also be deduced from the following identity:

For any complex number $z \neq 1$, we have

$$1 + z + z^2 = \frac{1 - z^3}{1 - z} \qquad \textbf{(2.16)}$$

The identity (2.16) can be immediately deduced from the following simple result:

For $z \neq 1$,

$$(1 - z)(1 + z + z^2) = 1 + z + z^2 - z - z^2 - z^3 = 1 - z^3.$$

If we set $z = \omega$, the second cube root of unity, then

$$1 + \omega + \omega^2 = \frac{1 - \omega^3}{1 - \omega} = 0, \text{ since } \omega^3 = 1.$$

If in relation (2.15), we set $n = 4$, $\rho = 1$ and $\psi = 0$, we get the four roots of unity. These four roots are:

$1, \cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}, \cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4}$ and $\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}$.

or, $1, \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}, \cos\frac{2\pi}{2} + i\sin\frac{2\pi}{2}$, and $\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$.

If we set $\omega = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$, then the third and the fourth roots are respectively $\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^2$ and $\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^3$. Thus the four roots are 1, ω, ω^2 and ω^3,

where $\omega = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$. Hence the four roots are 1, i, i^2 and i^3 or the four roots are 1, i, -1, and $-i$. It is easily seen from the values of the four roots that their sum is zero. The four fourth roots of unity form the vertices of a square all lying on the unit circle as shown in figure 2.9.

Remark

It may be noted that the symbol ω has been used in the preceding discussion to denote first a cube root of unity, and later a fourth root of unity and these two roots are different.

If we take $\rho = 1$ and $\psi = 0$ in relation (2.15), we get the nth roots of 1. These n roots are $1, \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$,

$\cos 2.\frac{2\pi}{n} + i\sin 2.\frac{2\pi}{n}$,

$\cos 3.\frac{2\pi}{n} + i\sin 3.\frac{2\pi}{n}$ ...,

and $\cos(n-1)\frac{2\pi}{n} + i\sin(n-1)\frac{2\pi}{n}$.

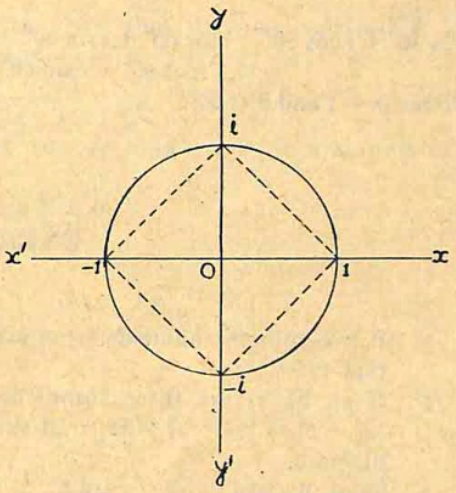

Fig. 2.9

Again, if we set $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, then the n roots are $1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$, by the Laplace-de Moivre formula. As stated in the beginning of this remark, the use of ω here is just for notational convenience.

It is easily seen that the nth roots of unity all lie on the unit circle and form the vertices of a regular polygon of n sides.

Example 2.6

Express $3 + 4i$ in polar form.

Solution

$$3 + 4i = 5\left(\frac{3}{5} + \frac{4}{5}i\right)$$

$$= 5(\cos\theta + i\sin\theta)$$

where θ is such that $\tan\theta = \frac{4}{3}$ as shown in figure 2.10. Note that $r = 5$ here.

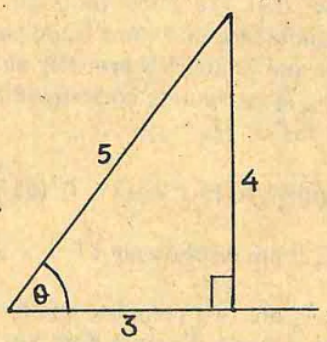

Fig. 2.10

Example 2.7

Express $\sin 30^0 + i \cos 30^0$ in polar form.

Solution

$\sin 30^0 = \cos 60^0$ and $\cos 30^0 = \sin 60^0$,

so that

$$\sin 30^0 + i \cos 30^0 = \cos 60^0 + i \sin 60^0$$
$$= 1 (\cos 60^0 + i \sin 60^0)$$

Hence, $r = 1$ and $\theta = 60^0$.

EXERCISE 2.2

1. If two complex numbers z_1, z_2 are such that $|z_1| = |z_2|$, is it then necessary that $z_1 = z_2$?
2. If z_1, z_2, z_3 are three complex numbers such that there exists a z with $|z_1 - z| = |z_2 - z| = |z_3 - z|$ show that z_1, z_2, z_3 lie on a circle in the Argand diagram.
3. Show that $\arg \bar{z} = 2\pi - \arg z$.

MISCELLANEOUS EXERCISE ON CHAPTER 2

1. Show that for $z \in \mathbf{C}$, $|z| = 0$ if and only if $z = 0$.
2. Show that the cube roots of unity lie on the unit circle and divide the circumference into three equal parts starting from $z = 1$.
3. State and establish a property analogous to that in 2 above for fourth roots.
4. If P_1, P_2 are points corresponding to z_1, $z_2 \in$, $\mathbf{C}$ in the Argand diagram, show that $P_1P_2 = |z_1 - z_2|$.
5. Simplify : (i) $(3 + 2i)(2 - i)$, (ii) $\dfrac{2 - 3i}{4 - i}$
6. If z_1, z_2 are respectively $1 - i$, $-2 + 4i$ respectively, find $\text{Im}\left(\dfrac{z_1 z_2}{\bar{z}_1}\right)$
7. If z_1, z_2 are two complex numbers, show that the straight line joining z_1 and z_2 in the Argand diagram does not pass through origin if and only if there exists no real numbers a and b, a, b, $\neq 0$, such that $az_1 + bz_2 = 0$.

8. If z is an arbitrary complex number, where does it lie in the Argand diagram under the constraint $|z + 2 - i| = 4$?
9. Find the square roots of $-15 - 8i$.
10. Describe the set of complex numbers z for which $\left|\frac{z-3}{z+3}\right| = 2$ geometrically.
11. If z_1, z_2, z_3, z_4 are complex numbers, show that they are vertices of a parallelogram in the Argand diagram if and only if $z_1 - z_2 - z_3 + z_4 = 0$.
12. Explain the fallacy: $-1 = i.i = \sqrt{-1}.\sqrt{-1}$
$= \sqrt{(-1)(-1)} = \sqrt{1} = 1$
13. Prove that any point z lies on the line joining z_1, z_2 if and only if $\arg\{(z - z_1)/(z_2 - z_1)\} = 0$
14. If $z = x + iy$, x,y real, prove that $|x| + |y| \leq \sqrt{2}|z|$.
15. Prove that
$\text{Re}\,(z_1z_2) = \text{Re}\,z_1\,\text{Re}\,z_2 - \text{Im}\,z_1\,\text{Im}\,z_2$
$\text{Im}\,(z_1z_2) = \text{Re}\,z_1\,\text{Im}\,z_2 + \text{Im}\,z_1\,\text{Re}\,z_2$

CHAPTER 3

QUADRATIC EQUATIONS

3.1 Solution of Quadratic Equations

You have studied polynomials in lower classes.
$3x^2 + 2x - 1$, $5x^3 - 3x + 2$, $4x^5 - 7x^4 - 6x^3 + 2x^2 + 3x + 6$ etc. are examples of polynomials. A function f defined by

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n,\ x \in \mathbf{R}$$

where $a_0, a_1 \ldots, a_n \in \mathbf{R}$ is called a *polynomial* of a real variable with real coefficients. If $a_n \neq 0$, it is said to be of degree n. If x is assumed to be a varying complex number and $a_0, a_1, \ldots, a_n \in \mathbf{C}$, so that the function $f: \mathbf{C} \to \mathbf{C}$, it is called a complex polynomial or a polynomial of a complex variable with complex coefficients. In the last chapter we were concerned with the polynomial $x^2 + 1$ and the corresponding equation $x^2 + 1 = 0$ which we noticed cannot be satisfied by any real x. This is a polynomial of second degree. Generally, we call a polynomial of the second degree a *quadratic polynomial*. Any equation $f(x) = 0$, where f is a quadratic polynomial, is called a *quadratic equation*. The general form of a quadratic equation is

$$ax^2 + bx + c = 0 \qquad \ldots(3.1)$$

where a, b, c are real numbers i.e. $a, b, c \in \mathbf{R}$. We will discuss in this chapter, only such quadratic equations, and we do not preclude, for the sake of convenience, one or more of a, b, c being zero. We are interested in solving equation (3.1) and examining the nature of the roots. We shall omit the trivial case $a = b = c = 0$ for which any $x \in \mathbf{R}$ is a solution. First, let us see when we can have a solution in $\mathbf{R}$. If $a = 0$, the equation (3.1) reduces to

$$bx + c = 0,$$

and we may assume that $b \neq 0$ (otherwise, c should also be 0). Then we have the solution

$$x = -\frac{c}{b}$$

as the unique solution of (3.1), which lies in **R**. Now, assume that $a \neq 0$, and rewrite (3.1) as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Adding and subtracting $\left(\frac{b}{2a}\right)^2$, the equation becomes

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2 = 0$$

or

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \qquad \text{...(3.2)}$$

If x were to be a real number, $x + \frac{b}{2a}$ should be a real number, since $\frac{b}{2a}$ is a real number (why?). In fact, x is real if and only if $x + \frac{b}{2a}$ is real. This would mean that x is real if and only if $\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$ is non-negative. Since $a^2 > 0$ this means that there exists a real solution for (3.1) if and only if

$$\Delta = b^2 - 4ac \geq 0.$$

Δ is known as the *discriminant* of the quadratic equation (3.1). For instance, it discriminates between the situation when x can have a real value to satisfy (3.1) and when it cannot. In case it has a solution which is a real number, it has in fact two such solutions given by

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

where $\sqrt{}$ stands for the unique non-negative square root of $b^2 - 4ac$ and $\pm$ stands for the possibility of taking this square root or its negative. Moreover, these are the only two solutions, viz.

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad \text{... (3.3)}$$

both of which are real in this case. If $\Delta \equiv b^2 - 4ac = 0$, we immediately conclude that both these solutions coincide with one another, i.e. the two roots are equal, each being equal to $-\frac{b}{2a}$.

Again, the two solutions coincide if and only if $\Delta = 0$, otherwise, we have two distinct real solutions of equation (3.1). We shall call the solutions of (3.1) its *roots*.

Let us now consider the case $\Delta<0$. From the earlier discussion it follows that the equation (3.1) does *not* have any real root in this case. If we set $x+\frac{b}{2a} = X$ in (3.2) we have

$$X^2 = \frac{b^2-4ac}{4a^2} = -\frac{4ac-b^2}{4a^2} \qquad \ldots (3.4)$$

where $\frac{4ac-b^2}{4a^2}$ is positive. i is a square root of -1 (See Chapter 2, Section 2.1) so that $-i$ is also a square root of -1. On taking square roots on both sides of (3.4) we get

$$X = \frac{\pm i\sqrt{4ac-b^2}}{2a}$$

Hence, $x+\frac{b}{2a} = \pm\frac{i\sqrt{4ac-b^2}}{2a}$

or $x = \frac{-b\pm i\sqrt{4ac-b^2}}{2a}$

Thus the two solutions of equation (3.1) are given by

$$x = \frac{-b+i\sqrt{4ac-b^2}}{2a}, \quad \frac{-b-i\sqrt{4ac-b^2}}{2a}$$

These two complex roots are evidently conjugate of each other. If we prefer to write $i\sqrt{4ac-b^2}$ as $\sqrt{b^2-4ac}$, we have the solutions

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} \qquad \ldots (3.5)$$

which is formally the same as (3.4). Thus, in any case (3.4) gives the two roots of the quadratic equation (3.1) in $\mathbf{C}$ and equation (3.1) is completely solvable in $\mathbf{C}$. Denote, for convenience, by α, β the roots of the equation (3.1) as given by (3.3). Then

$$\begin{aligned}(x-\alpha)(x-\beta) &= \left\{\left(x+\frac{b}{2a}\right)+\frac{\sqrt{b^2-4ac}}{2a}\right\}\times\left\{\left(x+\frac{b}{2a}\right)-\frac{\sqrt{b^2-4ac}}{2a}\right\}\\ &= \left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2} = x^2+\frac{b^2}{4a^2}+2x\frac{b}{2a}-\frac{b^2}{4a^2}+\frac{4ac}{4a^2}\\ &= x^2+\frac{bx}{a}+\frac{c}{a}\\ &= \frac{1}{a}(ax^2+bx+c)\end{aligned}$$

Thus we see that